GLOW Maths Hub Conference: MathsFest19

The Chase Hotel, Cheltenham, 14 March 2019

YesUCan Do Proof

- Mathematical argument, language and proof in AS and A level

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- Reformed Mathematics AS/A level Content requires students to demonstrate a wide range of overarching knowledge and skills, and that these must be applied, along with associated mathematical thinking and understanding, across the whole of the detailed Content.
- In respect of the first of these: OT1 Mathematical argument, language and proof three specific aims and objectives are that students must be encouraged to:
  - reason logically and recognise incorrect reasoning
  - generalise mathematically
  - construct mathematical proofs
- Aside from learning about different techniques of proof, and developing problem solving approaches, the main purpose of the reforms is to encourage students to develop logical thought, and to be able to provide clear mathematical arguments in support of a result, using correct mathematical notation and language. This is the primary purpose of OT1.

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OT1 Mathematical argument, language and proof

AS and A level mathematics specifications must use the mathematical notation set out in appendix A and must require students to recall the mathematical formulae and identities set out in appendix B.

	Knowledge/Skill
OT1.1	**[Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable]**
OT1.2	**[Understand and use mathematical language and syntax as set out in the content]**
OT1.3	**[Understand and use language and symbols associated with set theory, as set out in the content]** **[Apply to solutions of inequalities]** and probability
OT1.4	Understand and use the definition of a function; domain and range of functions
OT1.5	**[Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics]**

Detailed content statements

11. A level specifications in mathematics must include the following content. This, assessed in the context of the overarching themes, represents 100% of the content.

12. Content required for AS mathematics is shown in bold text within square brackets. This, assessed in the context of the AS overarching themes, represents 100% of the AS content.

A Proof

	Content
A1	**[Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion]** **[Disproof by counter example]** Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs)

Proof - direct

Prove that for all $n \in \mathbb{Z}$, if n is odd, then n^2 is odd.

Proof - direct

The sum of two odd numbers is even.

The product of two odd numbers is odd.

The sum of an even number and an odd number is odd.

The sum of two even numbers is even.

Proof - direct

The product of two consecutive integers is even.

The product of two *rational* numbers is always *rational*.

Proof - exhaustion

For all $n \in \mathbb{N}$, $n^3 - n$ is a multiple of 3.

Proof - exhaustion

For all $n \in \mathbb{N}$, if $n = m^3 - m$ for some $m \in \mathbb{N}$, then n is a multiple of 6.

Proof - exhaustion

The square of any natural number is of the form $3k$ or $3k+1$ for some $k \in \mathbb{N}$.

Prove that every integer that is a perfect cube is either a multiple of 9, or 1 less, or 1 more than a multiple of 9.

Proof – counterexample

Disprove the statement that: for all $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

Proof - contradiction

For all $n \in \mathbb{N}$, if n^2 is even, then n is even.

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Proof - contradiction

For all $n \in \mathbb{N}$, if n^2 is odd, then n is odd.

Proof - contradiction

For all $n \in \mathbb{N}$, if n^2 is a multiple of 3, then n is a multiple of 3.

Prove that for every $n \in \mathbb{N}$ $n^2 + 2$, is not divisible by 4.

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Proof – contradiction

$\sqrt{2}$ is irrational.

$\sqrt{3}$ is irrational.

$\sqrt{4}$ is irrational.

Proof - contradiction

Prove that $\sqrt[3]{2}$ is irrational.

Proof - contradiction

Suppose $a, b \in \mathbb{R}$ and $a \neq 0$. Prove that if a is rational and ab is irrational then b is irrational.

For all $a \in \mathbb{Q}$, and all irrational numbers b, then ab is irrational.

Proof - contradiction

For all $p \in \mathbb{Q}$, $p \neq 0$, and all irrationals q, pq is irrational.

For all $m, n \in Z$, if mn is even, then either m or n is even.

Proof - contradiction

For all $a \in \mathbb{Q}$ and all irrational numbers b, $a + b$ is irrational.

For all $m, n \in \mathbb{Z}$, if $m^2 + n^2$ is even, then $m + n$ is even.

Proof - contradiction

If $a, b \in \mathbb{Z}$ then $a^2 - 4b \neq 2$.

If $a, b \in \mathbb{Z}$ then $a^2 - 4b \neq 3$.

Proof - contradiction

No integer n exists such that $4n + 3$ is a square number.

Prove that for every $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4.

Proof - contradiction

Prove that for all $n \in \mathbb{N}$, if $4^n - 1$ is prime, then n is odd.

Proof - contradiction

For all $n \in \mathbb{N}$, if $2^n - 1$ is prime, then n is prime.

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Proof - contradiction

For all $n \in \mathbb{N}$, if $a^n - 1$ is prime, then $a = 2$ and n is prime.

Proof – miscellaneous

The sum of two consecutive odd numbers is equal to the difference of two square numbers.

Prove that for all $x \in [\frac{1}{2}\pi, \pi]$, $\sin x - \cos x \geq 1$.

Between any two *rational* numbers there is another *rational* number.

The product of 5 consecutive integers is a multiple of 120.

For all $n \in \mathbb{N}$, the sum of n consecutive natural numbers is:

(a) even if n is a multiple of 4;

(b) odd if n is even but *not* a multiple of 4;

(c) divisible by n if n is odd.

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Proof - miscellaneous

There exists *irrational* numbers a, b such that a^b is rational.

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Proof - miscellaneous

There exists *irrational* numbers a, b such that a^b is rational.

Hint: Consider $\sqrt{2}^{\sqrt{2}}$.

Proof - miscellaneous

Prove that there are no integer solutions of $m^2 - n^2 = 22$.

The product of two *irrational* numbers is always *irrational.*

Proof - counting

How many subsets, T, are there of a set, A, of n elements?

$$A = \{a_1, a_2, \ldots, a_{n-1}, a_n\}$$

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Proof - counting

number of elements, n	set	subsets	no. of subsets, N
1	$\{a\}$	$\{\},\{a\}$	2
2	$\{a,b\}$	$\{\},\{a\},\{b\},\{a,b\}$	4
3	$\{a,b,c\}$	.	.
.	.	.	.

Proof - counting

$$\left\{a_1, a_2, \ldots, a_{n-1}, a_n\right\}$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

$$(\alpha+\beta)^n = \binom{n}{0}\alpha^n\beta^0 + \binom{n}{1}\alpha^{n-1}\beta^1 + \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots + \binom{n}{n-1}\alpha^1\beta^{n-1} + \binom{n}{n}\alpha^0\beta^n$$

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Proof - counting

Given the set $A = \{a_1, a_2, \ldots, a_{n-1}, a_n\}$ and a subset T define $s(T)$ as a string of zeros and ones whose length is the number of elements in S constructed in the following way:

in the jth position of $s(T)$ put:

- 1 if $a_j \in T$
- 0 if $a_j \notin T$

Proof - counting

Which positive integers have exactly 3,4,5 positive divisors?

Proof - counting

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

$$\text{number of divisors } d = (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$$

$$d = 1$$
$$d = 2$$
$$d = 3$$
$$d = 4$$
$$d = 5$$

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Some links:

A mathematician's miscellany, and an apology

Further outreach materials

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Thank you

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